Physics Recreations:
The Rocket Equation

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1 Introduction

A *rocket* is a vehicle that propels itself through space by ejecting a propellant gas at high speed in a direction opposite the desired direction of motion. The German V-2 rocket was an early example, as were the United States rockets such as Juno, Redstone, Agena, and Saturn. The largest and most powerful rocket ever used is the United States Saturn V Moon rocket, which took the Apollo astronauts to the Moon in the 1960s and 1970s.

In order to place a spacecraft into low-Earth orbit, a rocket must accelerate its payload from rest to a speed of about 17,000 miles per hour. In order to reach this speed, most of the rocket's mass must be fuel. The amount of fuel required for a given mass of payload is governed by the *rocket equation*, which will be derived here.

Some critics of early space exploration claimed that rockets would not be able to travel in space because "they would have nothing to push against." As we'll see here, such arguments are silly—one needs only to make use of the conservation of momentum to show that rockets can work in space.

2 Derivation of the Equation

Let's now derive the rocket equation. Given a rocket of mass m, we will wish to find an equation that tells us how much fuel (propellant) is required to change the rocket's speed by an amount Δv. The complication here is that the rocket loses mass as it expels propellant, so we need to allow for that.

Suppose that at an initial time $t = 0$, a rocket has velocity v and total mass m, including propellant mass. The total momentum of the rocket and propellant at time $t = 0$ is therefore mv.

Now let's look at the situation an instant later, at time $t = dt$. Let dm be the (negative) change in mass of the rocket due to the expulsion of propellant, and let dv be the corresponding (positive) change in the velocity of the rocket. Then at time $t = dt$, a mass of propellant $-dm$ is expelled at velocity $v - v_p$. (The rocket is moving at velocity v with respect to the Earth, the propellant is moving at speed $-v_p$ *relative to the rocket*, and so the velocity of the propellant relative to the Earth is $v - v_p$.) This expulsion of propellant will cause the rocket to then have mass $m + dm$ and velocity $v + dv$. The total momentum of the system at $t = dt$ is then the sum of the rocket and propellant momenta, $(m + dm)(v + dv) + (v - v_p)(-dm)$. By conservation of momentum, the momentum of the system at time $t = 0$ must equal the momentum at time $t = dt$:

$$\begin{aligned} mv &= (m + dm)(v + dv) + (v - v_p)(-dm) && (1) \\ &= mv + v\,dm + m\,dv + dm\,dv - v\,dm + v_p\,dm && (2) \\ & && (3) \end{aligned}$$

Now the two mv terms cancel, the two $v\,dm$ terms cancel, and the term $dm\,dv$ is a second-order differential, which can also be cancelled. We're then left with

$$\begin{aligned} 0 &= m\,dv + v_p\,dm && (4) \\ m\,dv &= -v_p\,dm && (5) \\ dv &= -v_p\,\frac{dm}{m} && (6) \end{aligned}$$

Now let the rocket burn all its propellant. The rocket's velocity will change by a total amount Δv and its mass will change from m to its empty mass m_e. Integrating Eq. (6) over the entire propellant burn, we find

$$\int_v^{v+\Delta v} dv = -v_p \int_m^{m_e} \frac{dm}{m} \tag{7}$$

Or, evaluating the integrals,

$$\Delta v = -v_p \ln \frac{m_e}{m} \tag{8}$$

or

$$\boxed{\Delta v = v_p \ln \frac{m}{m_e}} \tag{9}$$

Eq. (9) is called the *rocket equation*. It relates the fueled and empty masses of the rocket and the velocity of the propellant to the total change in velocity of the rocket.

3 Mass fraction

The fraction of the total initial mass m that is propellant is

$$\frac{\text{propellant mass}}{\text{total initial mass}} = \frac{m - m_e}{m} = 1 - \frac{m_e}{m}. \tag{10}$$

Solving Eq. 8 for this fraction, we find

$$1 - \frac{m_e}{m} = 1 - e^{-\Delta v / v_p} \tag{11}$$

Eq. (11) tells what fraction of the rocket's total mass must be fuel in order to achieve a desired change in rocket velocity Δv.

4 Example

Let's take as an example the launch of a rocket from the Earth's surface to low-Earth orbit. In this case, the rocket's velocity will need to change by an amount Δv = 17,000 mph, or about 7600 m/s. Let's say we have a rocket that can expel propellant with a speed $v_p = 4000$ m/s. Then by Eq. (11),

$$1 - \frac{m_e}{m} = 1 - e^{-\Delta v / v_p} = 0.85, \tag{12}$$

so 85% of the rocket's initial mass must be propellant.

5 Staging

In practice, it is found that it can be more efficient to launch rockets in *stages*, where part of the rocket structure drops away when it is no longer needed, thus decreasing the amount of mass that needs to be placed in orbit. For example, the Saturn V rocket had three stages. The large lower first stage contained a large fuel tank and large engines. When all the fuel contained in that stage had been spent, the entire first stage separated and dropped away, and a smaller second stage was ignited. When all the second-stage fuel was spent, it too separated and and dropped away, and the third stage engine ignited, which placed the spacecraft into Earth orbit. This staged approach requires much less fuel than launching the entire Saturn V rocket into orbit.