The Arithmetic-Geometric Mean

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Averages

Suppose you want to compute the average of a set of numbers. There are a number of ways of doing this; for example, if you arrange the numbers in order in a list, the value in the middle of the list is called the median average. You may have a set of numbers in which the same number occurs more than once; the value that occurs most often is called the mode average. The most common method, though, is to compute the sum of all the numbers and divide by how many numbers there are; this is called the arithmetic mean.

For two numbers \(x\) and \(y\), the arithmetic mean \(a\) is given by

\[
a = \frac{x + y}{2}.
\]

Still another kind of average is called the geometric mean. The geometric mean \(g\) of two numbers \(x\) and \(y\) is the square root of their product:

\[
g = \sqrt{xy}.
\]

Arithmetic-Geometric Mean

A lesser-known type of average combines the arithmetic and geometric means together. Suppose we’re given two numbers \(x\) and \(y\). Then compute the arithmetic and geometric means of \(x\) and \(y\):

\[
a_1 = \frac{x + y}{2}, \quad g_1 = \sqrt{xy}.
\]

Now compute the arithmetic and geometric means of these two means:

\[
a_2 = \frac{a_1 + g_1}{2}, \quad g_2 = \sqrt{a_1g_1}.
\]

And now compute the arithmetic and geometric means of these two means:

\[
a_3 = \frac{a_2 + g_2}{2}, \quad g_3 = \sqrt{a_2g_2}.
\]
And so on. We iterate, repeating this process over and over,

\[ a_{n+1} = \frac{a_n + g_n}{2} \] \hspace{1cm} (9)

\[ g_{n+1} = \sqrt{a_n g_n} \] \hspace{1cm} (10)

and we discover that the two means will converge to the same number. The number to which these means converge is called the \textit{arithmetic-geometric mean} of \( x \) and \( y \).

**Example**

Suppose our two numbers are 7 and 12. Then their arithmetic mean is

\[ a_1 = \frac{7 + 12}{2} = 9.5. \] \hspace{1cm} (11)

Their geometric mean is

\[ g_1 = \sqrt{7 \times 12} = \sqrt{84} = 9.165151. \] \hspace{1cm} (12)

Now let’s find the arithmetic-geometric mean of 7 and 12 by running these two means through the iteration process (Equations 9 and 10). We find

\[
\begin{align*}
  a_1 &= 9.500000 \quad g_1 = 9.165151 \\
  a_2 &= 9.332576 \quad g_2 = 9.331074 \\
  a_3 &= 9.331074 \quad g_3 = 9.331074
\end{align*}
\]

So the two values have quickly converged to the arithmetic-geometric mean, which is 9.331074.

**The Simple Plane Pendulum**

Where would we use the arithmetic-geometric mean? One application is described in a recent paper by Adlaj [1], where it used to calculate the \textit{exact} period of a simple plane pendulum of length \( L \) and amplitude \( \theta_0 \). The appendices to the course notes give a formula for this period in terms of an infinite series:

\[
T = 2\pi \sqrt{\frac{L}{g}} \left\{ 1 + \sum_{n=1}^{\infty} \frac{(2n)!}{2^{2n}(n!)^2} \frac{\theta_0^2}{2} \sin^2 \left( \frac{\theta_0}{2} \right) \right\}. \] \hspace{1cm} (13)

According to Adlaj, the exact period of a simple plane pendulum may be calculated more efficiently using the arithmetic-geometric mean, by means of the formula

\[
T = 2\pi \sqrt{\frac{L}{g}} \times \frac{1}{a_{\text{agm}}(1, \cos(\theta_0/2))} \] \hspace{1cm} (14)

where \( a_{\text{agm}}(x, y) \) denotes the arithmetic-geometric mean of \( x \) and \( y \). The factor on the right involving the arithmetic-geometric mean is essentially a correction factor that corrects the small-angle approximation for the period \( (T \approx 2\pi \sqrt{L/g}) \) to the exact value.

Surprisingly, Eq. (14) is \textit{not} just an approximation of Eq. (13). In theory, both formulæ return the same results, although Eq. (14) is actually numerically better behaved for amplitudes approaching \( \theta_0 = 180^\circ \).
Example

Suppose we have a simple plane pendulum with length $L = 1.000$ meter, and with amplitude $\theta_0 = 60^\circ$. Using the series expansion (Eq. (13)) we compute the period $T = 2.153973$ sec.

Now find the period of the same pendulum again, but this time using the arithmetic-geometric mean, Eq. (14). We find $\text{agm}(1, \cos(60^\circ/2)) = 0.931808$, which gives $T = 2.153973$ sec — the same result we got using the series expansion.

The arithmetic-geometric mean formula for the pendulum period (Eq. 14) has several advantages over the series expansion (Eq. 13):

- The formula is much simpler.
- It is computationally more efficient.
- It is numerically better behaved for amplitudes near $\theta_0 = 180^\circ$.

References