The World Ceres

Dr. D.G. Simpson
Department of Physical Sciences and Engineering
Prince George's Community College

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1 Introduction

Ceres is the largest asteroid in the *asteroid belt*—a population of small rocky bodies that occupy the region of the Solar System between the orbits of Mars and Jupiter. It is about the size of Texas, and is the only asteroid large enough that its mass pulls it into a roughly spherical shape. Ceres is today technically classified as a *dwarf planet*, and is the only dwarf planet in the asteroid belt.

Recent studies [1] suggest that the interior of Ceres is *differentiated*—that is, rather than being uniform and homogeneous, it consists of a dense inner core surrounded by a lighter exterior. How could we possibly know that, when there's no way to look inside Ceres and see its interior? The answer lies in observing Ceres' shape and rotation: a rotating fluid body is expected to become flattened by a certain amount (becoming flatter the faster it rotates), and the observed amount of Ceres' flattening is inconsistent with its rotation rate if Ceres is assumed to be uniform. Therefore, we conclude that its interior cannot be uniform, i.e. it must be differentiated.

2 The Details: Ceres

It can be shown that a uniform, homogeneous, spherical, fluid body will, when it is set rotating, assume the shape of an oblate ellipsoid—that is, an ellipsoid that has its short axis is along the axis of rotation, and two equal long axes toward the equator. The relationship between the shape of the rotating ellipsoid and its rotation rate is given by *Maclaurin's formula*[1] [2],

$$\Omega = \left\{ \pi G \rho \left[\frac{(1-e^2)^{1/2}}{e^3} 2(3-2e^2)(\sin^{-1} e) - \frac{6}{e^2}(1-e^2) \right] \right\}^{1/2}. \tag{1}$$

Here Ω is the rotation rate (in radians per second), G is Newton's gravitational constant, ρ is the (constant) density of the body, and e is the eccentricity of the body,

$$e = \sqrt{1 - \frac{b^2}{a^2}} \tag{2}$$

[1] Note the inverse sine function in Maclaurin's formula. It is assumed that this function returns an angle in *radians*.

where a is the semi-major axis (equatorial radius) and b is the semi-minor axis (polar radius) of the body. The eccentricity e describes how "flattened" the ellipsoid is: $e = 0$ for a perfect sphere, and e approaches 1 for a very flat, pancake-shaped ellipsoid.

Let's try plugging in some numbers. From observations, we find that Ceres has:

- Semi-major axis (equatorial radius): $a = 487.3$ km
- Semi-minor axis (polar radius): $b = 454.7$ km
- Mass: $m = 9.395 \times 10^{20}$ kg
- Observed rotation period: $T = 9.075$ hours

From these numbers, we can find the volume of Ceres, using the formula for the volume of an oblate ellipsoid:

$$V = \frac{4}{3}\pi a^2 b = 4.523 \times 10^{17}\ \mathrm{m}^3. \tag{3}$$

From the volume and mass, we can find the density:

$$\rho = \frac{m}{V} = 2077\ \mathrm{kg/m}^3. \tag{4}$$

The eccentricity of Ceres is found from Eq. (2):

$$e = \sqrt{1 - \frac{b^2}{a^2}}\ =\ 0.3596. \tag{5}$$

Using these values for ρ and e and the gravitational constant $G = 6.67384 \times 10^{-11}$ m^3 kg^{-1} s^{-2}, we find, using Maclaurin's formula (1), the rotation rate that would be expected to produce this eccentricity *if* Ceres were uniform and homgeneous throughout:

$$\Omega = 1.7489 \times 10^{-4}\ \mathrm{rad/s} \tag{6}$$

The corresponding expected rotation period is then

$$T = \frac{2\pi}{\Omega} = 35{,}926\ \mathrm{sec} = 9.979\ \mathrm{hours}. \tag{7}$$

But the observed rotation rate is significantly faster, with period $T = 9.075$ hours. In other words, if Ceres were a uniform, homogeneous body, then we would expect it to be rotating more slowly than it actually is: the observed flattening is inconsistent with the flattening that would be expected from a uniform body. We therefore infer that Ceres is *not* uniform inside—it is differentiated.

As described in Ref. [1], scientists can take this idea a bit further. One can create mathematical models of differentiated ellipsoids that have cores with various sizes and densities that are consistent with the observed total mass, then calculate how flat they become at the observed rotation rate. This method allows scientists to determine some plausible (if not necessarily unique) models for the structure of the interior of Ceres.

3 Another Example: Earth

As another example, we can do the same analysis for the Earth, for which we have:

- Semi-major axis (equatorial radius): $a = 6378.1366$ km
- Semi-minor axis (polar radius): $b = 6356.7519$ km
- Mass: $m = 5.9726 \times 10^{24}$ kg
- Observed rotation period:[2] $T = 23.9345$ hours

Proceeding as before, we need to find the density ρ and eccentricity e of series, then substitute into Maclaurin's formula. The volume of the Earth is

$$V = \frac{4}{3}\pi a^2 b = 1.0832 \times 10^{21}\ \mathrm{m}^3. \tag{8}$$

From the volume and mass, we can find the density of the Earth:

$$\rho = \frac{m}{V} = 5513.8\,\mathrm{kg/m}^3. \tag{9}$$

The eccentricity of the Earth is found from Eq. (2):

$$e = \sqrt{1 - \frac{b^2}{a^2}} \;=\; 0.081819. \tag{10}$$

Using these values for ρ and e, we find, using Maclaurin's formula (1), the rotation rate that would be expected to produce this eccentricity *if* the Earth were uniform and homgeneous throughout:

$$\Omega = 3.9448 \times 10^{-5}\ \mathrm{rad/s} \tag{11}$$

which corresponds to a rotation period of

$$T = \frac{2\pi}{\Omega} = 159{,}277\,\mathrm{sec} = 44.243\,\mathrm{hours}. \tag{12}$$

But the observed rotation rate is significantly faster, with period of just $T = 23.9349$ hours. In other words, the Earth is not as flat as would be expected based on its rotation period, *if* it were homogeneous and had a constant density throughout. We therefore infer that the Earth is, like Ceres, differentiated. This has been confirmed for the Earth by making observations of seismic waves produced by earthquakes: geologists can infer the properties of the Earth's interior by studying what happens to these seismic waves as they travel through the Earth's interior.

References

[1] P.C. Thomas, J.Wm. Parker, L.A. McFadden, C.T. Russell, S.A. Stern, M.V. Sykes, and E.F. Young, Differentiation of the asteroid Ceres as revealed by its shape. *Nature*, **437**, 224–226 (September 2005).

[2] S. Chandrasekhar, *Ellipsoidal Figures of Equilibrium*. Yale University Press, 1969.

[2] Not 24 hours like you might expect. It takes 24 hours for the *Sun* to return to the same position in the sky each day, but what we want is the rotation period of the Earth relative to the fixed *stars*. This is called the *sidereal rotation rate*, and is about 4 minutes short of 24 hours.