Special Relativity

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1 Introduction

The classical mechanics described by Sir Isaac Newton begins to break down at very high velocities, i.e. at velocities near the speed of light $c = 299{,}792.458$ km/s. For bodies moving at a significant fraction of the speed of light, Newton's mechanics needs to be modified. The necessary modifications were developed by physicist Albert Einstein in the early 20th century.

2 Postulates

Einstein discovered that the necessary modifications to Newtonian mechanics could be derived by assuming two postulates:

1. Absolute uniform motion cannot be detected.
2. The speed of light is independent of the motion of the source.

The first postulate says that all motion is relative—that there is no reference frame that all observers can agree to be absolutely at rest. The second postulate says that light does not obey the usual laws of velocity addition. For example, if someone is moving toward you at 99% of the speed of light and turns on a flashlight in your direction, you will measure the light's speed to be the same as if that person were at rest.

Although these postulates seem quite reasonable, they lead to some surprising consequences. Let's examine a few of those consequences.

3 Time Dilation

It turns out that one consequence of Einstein's postulates is that time runs more slowly for someone moving relative to you; this effect is called *time dilation*. If someone is moving at speed v relative to you, then their clocks will run slower than yours. If a clock measures a time interval Δt_0 when it's at rest, then when it's moving at a speed v relative to you, you will measure that time interval to be longer by a factor γ:

$$\Delta t = \gamma\, \Delta t_0, \tag{1}$$

where Δt is the time interval measured by the moving clock, Δt_0 is the time interval measured on the clock when it's at rest, and γ is an abbreviation for the factor

$$\gamma \equiv \frac{1}{\sqrt{1 - v^2/c^2}}. \tag{2}$$

(Note that $\gamma \geq 1$.) The time interval Δt_0, measured when you're at rest with respect to the clock, is called the *proper time*.

This effect means that time travel is possible—at least time travel into the future. One simply builds a spacecraft and travels close to the speed of light, then turns around and returns to Earth. (It is not clear whether time travel into the past is possible, but it might be possible under Einstein's *general* theory of relativity.)

4 Length Contraction

Another consequence of the postulates is that a moving body will appear to be shortened in the direction of motion; this effect is called *length contraction*. The length of a moving body will appear to be shortened by this same factor of γ:

$$L = \frac{L_0}{\gamma} \tag{3}$$

Here L_0 is the length of the body when it is at rest, and is called the *proper length*. Since $\gamma \geq 1$, the moving body will be shorter when it is moving.

5 An Example

As an example, let's imagine that a spacecraft is launched at high speed relative to the nearest star, Alpha Centauri (which is about 4 light-years away). The ship travels at 80% of the speed of light during the trip. From Earth, we see that the whole trip takes 5 years. We also see the astronaut's clocks running more slowly than ours by a factor of $\gamma = 2.78$, so that when the astronauts arrive, they are only 1.8 years older.

What do the astronauts see from their point of view on the spacecraft? Their clocks run at what seems a normal rate for them, but they see that the *distance* to Alpha Centauri has been length-contracted by a factor of $\gamma = 2.78$. They're traveling at a speed of $0.80c$, but they only have to travel a distance of (4 light-years)/$\gamma =$ 1.44 light-years. When they arrive at Alpha Centauri, they're older by (1.44 light-years)/$0.80c = 1.8$ years.

In summary, observers on Earth see the astronaut's clocks moving more slowly, but the astronauts have to travel the full 4 light-years. The astronauts see their clocks moving at normal speed, but the distance they have to travel is shorter. All observers agree that the astronauts are only 1.8 years older when they arrive.

6 Momentum

In Newton's classical mechanics, momentum is $\mathbf{p} = m\mathbf{v}$. Under special relativity, this is modified to be

$$\mathbf{p} = \gamma m \mathbf{v}. \tag{4}$$

Relativistically, it is this definition of momentum that is conserved. Newton's Second Law in the form $\mathbf{F} = m\mathbf{a}$ is no longer valid under special relativity, but Newton's original form $\mathbf{F} = d\mathbf{p}/dt$ is still valid, using this definition of momentum $\mathbf{p}$.

Notice that as $v \to c$, we have $\gamma \to \infty$ (by Eq. (2)), and so momentum $p \to \infty$. As a body goes faster, its momentum increases in such a way that it becomes increasingly difficult to make it go even faster. This means that it is not possible for a body to move faster than the speed of light in vacuum, c.

7 Addition of Velocities

Let's suppose that we have two bodies moving in one dimension. The first is moving at speed u, and the second is moving at speed v. What is the speed of the second relative to the first? In other words, what will you measure as the speed of the second body if you're sitting on the first body?

In classical Newtonian mechanics, the speed w of the second body relative to the first is simply

$$w = v - u. \tag{5}$$

For example, if the first body is moving to the right with speed $u = 10$ m/s, and the second body is moving toward it to the left with speed $v = -20$ m/s, then an observer on the first body will see the second body moving toward it with a speed of $w = 30$ m/s.

In the special theory of relativity, this seemingly self-evident equation for adding velocities must be modified as follows:

$$w = \frac{v - u}{1 - uv/c^2}. \tag{6}$$

This reduces to Eq. (4) unless the speeds involved are near the speed of light. For the above example, where $u = 10$ m/s and $v = -20$ m/s, Eq. (5) gives $w = 29.99999999999993324$ m/s, rather than $w = 30$ m/s given by Eq. (4). As you can see, for many applications, the difference between the classical formula (4) and the exact relativistic formula (5) is not enough to justify the extra complexity of using the relativistic formula.

But for speeds near the speed of light, using the relativistic formula is important. For example, if $u = 0.99c$ and $v = -0.99c$, then the classical formula of Eq. (4) would give $w = 1.98c > c$, in violation of special relativity; but using the exact expression in Eq. (5) gives the correct answer, $w = 0.9999494975c$.

Eq. (5) makes it impossible for the the relative speeds to be greater than the speed of light c. In the extreme case $u = c$ and $v = -c$, Eq. (5) gives $w = c$, in agreement with the Einstein's second postulate.

8 Energy

8.1 Rest Energy

Einstein showed that mass is a form of energy, as shown by his most famous equation,

$$E_0 = mc^2. \tag{7}$$

E_0 is called the *rest energy* of the particle of mass m. The clearest illustration of this formula is the mutual annihilation of matter and *antimatter* (a kind of mirror-image of ordinary matter). When a particle of matter collides with a particle of antimatter, the mass of the two particles is converted completely to energy, the amount of energy liberated being given by Eq. (7).

As examples, the rest energy of the electron is 511 keV, and the rest energy of the proton is 938 MeV. (1 eV is one *electron volt*, and is equal to $1.60217653 \times 10^{-19}$ J.)

8.2 Kinetic Energy

In classical Newtonian mechanics, the kinetic energy is given by $K = mv^2/2$. The relativistic version of this equation is

$$K = (\gamma - 1)mc^2. \tag{8}$$

It is not obvious that this reduces to the classical expression until we expand γ into a Taylor series:

$$\gamma = \left(1 - \frac{v^2}{c^2}\right)^{-1/2} = 1 + \frac{1}{2}\frac{v^2}{c^2} + \frac{3}{8}\frac{v^4}{c^4} + \frac{5}{16}\frac{v^6}{c^6} + \frac{35}{128}\frac{v^8}{c^8} + \frac{63}{256}\frac{v^{10}}{c^{10}} + \frac{231}{1024}\frac{v^{12}}{c^{12}} + \cdots \tag{9}$$

Substituting this series expansion for γ into Eq. (8), we get

$$K = \frac{1}{2}mv^2 + \frac{3}{8}m\frac{v^4}{c^2} + \frac{5}{16}m\frac{v^6}{c^4} + \frac{35}{128}m\frac{v^8}{c^6} + \frac{63}{256}m\frac{v^{10}}{c^8} + \frac{231}{1024}m\frac{v^{12}}{c^{10}} + \cdots \tag{10}$$

Unless the speed v is near the speed of light c, all but the first term on the right will be very small and can be neglected, leaving the classical equation.

8.3 Total Energy

If the only forms of energy present are the rest energy E_0 and the kinetic energy K, then the total energy E will be the sum of these:

$$E = E_0 + K = \gamma mc^2. \tag{11}$$

It is often useful to know the total energy of a particle in terms of its momentum p rather than its velocity v. It can be shown that the total energy is given in terms of momentum by

$$E^2 = (pc)^2 + (mc^2)^2. \tag{12}$$

In the case where the total energy is much larger than the rest energy ($E \gg E_0$), we may neglect the second term on the right, and use

$$E \approx pc. \tag{13}$$