

Gauss's Law for Gravity

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Newton's Law of Gravity

Newton's law of gravity gives the force F between two point masses M and m , separated by a distance r :

$$F = G \frac{Mm}{r^2}, \quad (1)$$

where G is the universal gravitational constant, $6.67428 \times 10^{-11} \text{ m}^3 \text{ kg}^{-1} \text{ s}^{-2}$. By Newton's second law, force is related to acceleration by

$$F = ma. \quad (2)$$

Equating equations (1) and (2), we can solve for the acceleration a due to the gravity of a point charge:

$$a = g = \frac{GM}{r^2}. \quad (3)$$

Newton was able to show that the gravitational field due to a uniform sphere of mass is the same as if the sphere were a point mass located at the sphere's center of mass. (Newton invented the calculus in the process!) So, for example, we can use Equation (3) to find the acceleration due to gravity at the surface of the spherical Earth:

$$g = \frac{GM_e}{R_e^2}, \quad (4)$$

where M_e is the mass of the Earth, and R_e is the radius of the Earth. Substituting the known values of GM_e and R_e , we find

$$g = \frac{3.986005 \times 10^{14} \text{ m}^3 \text{ s}^{-2}}{(6.378140 \times 10^6 \text{ m})^2} = 9.80 \text{ m s}^{-2}, \quad (5)$$

which is the familiar value.

The Gaussian Formulation of Newtonian Gravity

An alternative formulation of Newtonian gravity is Gauss's Law for gravity. It states that the acceleration g due to gravity of a mass m (not necessarily a point mass) is given by

$$\oint_S \mathbf{g} \cdot \mathbf{n} dA = -4\pi Gm \quad (6)$$

This equation requires a bit of explanation. The circled integral sign indicates an area integral evaluated over a *closed* surface S . A closed surface may be a sphere, cube, cylinder, or some irregular shape—any closed surface that has a well-defined “inside” and “outside.” The integral is an area integral: we imagine that the surface S is divided into many infinitesimal squares, each of which has area dA . Performing the integral means summing up the integrand times dA over the entire closed surface S .

The vector \mathbf{g} is the acceleration due to gravity, as a *vector*. The vector always points *toward* the mass.

The vector \mathbf{n} is a *unit vector*, perpendicular (“normal”) to the surface S , and pointing *outward* from S .

On the right-hand side of Equation (6), we find familiar constants (π and G), along with mass m . Here m is the total mass *inside* surface S . It doesn’t matter what shape the mass m is, or how it is distributed; m is just the total mass inside surface S .

So Gauss’s law for gravity says this: we’re given some mass m , which may be of some arbitrary shape. Now imagine constructing an *imaginary* surface S around mass m (a sphere, or any other closed shape). Divide surface S into many infinitesimally small squares, each of which has area dA . At each square, draw a unit normal vector \mathbf{n} that is perpendicular to the surface at that square’s location, and which is pointing outward from S . Let \mathbf{g} be the acceleration due to gravity at that square. If we take the dot product of \mathbf{g} and \mathbf{n} at that square, multiply by the area of the square dA , then sum up all of those products for all the squares making up surface S , then the result will be $-4\pi G$ times the total mass enclosed by S .

This law applies in general, but in practice it is most useful for finding the acceleration to to gravity g due to a highly symmetrical mass distribution (a point, sphere, line, cylinder, or plane of mass). In these cases, the integral is particularly easy to evaluate, and we can easily solve for g in just a few steps.

Point Mass

For example, let’s use Gauss’s law to find the acceleration g due to the gravity of a *point* mass m . (The result should be Eq. 3.) We begin with a point mass m sitting in space. We now need to construct an imaginary closed surface S surrounding m . While in theory any surface would do, we should pick a surface that will make the integral easy to evaluate. Such a surface should have these properties:

1. The gravitational acceleration g should be either perpendicular or parallel to S everywhere.
2. The gravitational acceleration g should have the same value everywhere on S . (Or it may be zero on some parts of S .)
3. The surface S should pass through the point at which you wish to calculate the acceleration due to gravity.

If we can find a surface S that has these properties, the integral will be very simple to evaluate. For the point mass, we will choose S to be a *sphere* of radius r centered on mass m . Since we know \mathbf{g} points radially inward toward mass m , it is clear that \mathbf{g} will be perpendicular to S everywhere. Also, by symmetry, it is not hard to see that g will have the same value everywhere on S .

Having chosen a surface S , let us now apply Gauss’s law for gravity. The law states that

$$\oint_S \mathbf{g} \cdot \mathbf{n} \, dA = -4\pi Gm. \quad (7)$$

Now everywhere on the sphere S , $\mathbf{g} \cdot \mathbf{n} = -g$ (since \mathbf{g} and \mathbf{n} are anti-parallel— \mathbf{g} points inward, and \mathbf{n} points outward). Since $-g$ is a constant, Eq. (7) becomes

$$-g \oint_S dA = -4\pi Gm. \quad (8)$$

Now the integral is very simple: it is just dA integrated over the surface of a sphere, so it's just the area of a sphere:

$$\oint_S dA = 4\pi r^2. \quad (9)$$

Equation (8) is then just

$$-g(4\pi r^2) = -4\pi Gm, \quad (10)$$

or (cancelling -4π on both sides)

$$\boxed{g = \frac{Gm}{r^2}} \quad (11)$$

in agreement with Eq. (3) from Newton's law.

Line of Mass

The Gaussian formulation allows you to easily calculate the gravitational field due to a few other shapes. For example, suppose you have an infinitely long *line* of mass, having linear mass density λ (kilograms per meter), and you wish to calculate the acceleration g due to the gravity of the line mass at a perpendicular distance r of the mass. The appropriate imaginary "Gaussian surface" S in this case is a cylinder of length L and radius r , whose axis lies along the line mass. In this case, everywhere along the curved surface of cylinder S , the gravitational acceleration \mathbf{g} (pointing radially inward) is anti-parallel to the outward normal unit vector \mathbf{n} . Everywhere along the flat ends of the cylinder S , the gravitational acceleration \mathbf{g} is *perpendicular* to the outward normal vector \mathbf{n} , so that on the ends, $\mathbf{g} \cdot \mathbf{n} = 0$, and the ends contribute nothing to the integral. We therefore need only consider the curved surface of cylinder S .

Now apply Gauss's law:

$$\oint_S \mathbf{g} \cdot \mathbf{n} dA = -4\pi Gm. \quad (12)$$

Since \mathbf{g} is anti-parallel to \mathbf{n} along the curved surface of cylinder S , we have $\mathbf{g} \cdot \mathbf{n} = -g$ there. Bringing this constant outside the integral, we get

$$-g \oint_S dA = -4\pi Gm. \quad (13)$$

The integral is just the area of a cylinder:

$$\oint_S dA = 2\pi rL, \quad (14)$$

so Eq. (13) becomes

$$-g(2\pi rL) = -4\pi Gm. \quad (15)$$

Now m is the total mass enclosed by surface S . This is a segment of length L and density λ , so it has mass λL . This gives

$$-g(2\pi rL) = -4\pi G(\lambda L). \quad (16)$$

Cancelling $-2\pi L$ on both sides gives

$$\boxed{g = \frac{2G\lambda}{r}} \quad (17)$$

Plane of Mass

In addition to spherical and cylindrical symmetry, this technique may also be applied to plane symmetry. Imagine that you have an infinite *plane* of mass, having area mass density σ (kilograms per square meter), and you wish to calculate the acceleration g due to the gravity of the plane at a perpendicular distance r from the plane. The approach is similar to the previous cases: draw an imaginary closed “Gaussian surface,” write down Gauss’s law for gravity, evaluate the integral, and solve for the acceleration g .

In this case, the appropriate Gaussian surface S is a “pillbox” shape—a short cylinder whose flat faces (of area A) are parallel to the plane of mass. In this case, everywhere along the curved surface of S , the gravitational acceleration \mathbf{g} is *perpendicular* to the outward normal unit vector \mathbf{n} , so the curved sides of S contribute nothing to the integral. Only the flat ends of the pillbox-shaped surface S contribute to the integral. On each end, \mathbf{g} is anti-parallel to \mathbf{n} , so $\mathbf{g} \cdot \mathbf{n} = -g$ on the ends.

Now apply Gauss’s law to this situation:

$$\oint_S \mathbf{g} \cdot \mathbf{n} \, dA = -4\pi Gm. \quad (18)$$

Here the integral needs only to be evaluated over the two flat ends of S . Since $\mathbf{g} \cdot \mathbf{n} = -g$, we can bring $-g$ outside the integral to get

$$-g \oint_S dA = -4\pi Gm. \quad (19)$$

The integral in this case is just the area of the two ends of the cylinder, $2A$ (one circle of area A from each end). This gives

$$-g(2A) = -4\pi Gm. \quad (20)$$

Now let’s look at the right-hand side of this equation. The mass m is the total amount of mass enclosed by surface S . Surface S is sort of a “cookie cutter” that punches a circle of area A out of the plane. The mass enclosed by S is a circle of area A and density σ , so it has mass σA . Then Eq. (20) becomes

$$-g(2A) = -4\pi G(\sigma A) \quad (21)$$

Cancelling $-2A$ on both sides, we get

$$\boxed{g = 2\pi G\sigma} \quad (22)$$

Note that this is a constant: the acceleration due to gravity of an infinite plane of mass is independent of the distance from the plane!

In his science fiction novel *2010: Odyssey Two*, author Arthur C. Clarke describes a large rectangular slab that has been build by an alien race and placed in orbit around Jupiter. Astronauts are able to calculate the mass of the slab by placing a small spacecraft near the center of the large face and timing it to see how long it takes to fall to the surface of the slab. By approximating the slab as an infinite plane, they use Eq. (22) to find the acceleration; from that and the falling time, they can calculate the mass. (Actually, Dr. Clarke got the wrong answer in the book. You may want to find the book and see if you can calculate the correct answer.)

Gauss’s Law for Electrostatics

You will find the techniques described here will appear again in your study of electricity and magnetism. Classical electricity and magnetism is described by four equations called *Maxwell’s equations*; one of these is *Gauss’s Law*, and describes the electric field \mathbf{E} produced by an electric charge q :

$$\boxed{\oint_S \mathbf{E} \cdot \mathbf{n} \, dA = \frac{q}{\epsilon_0}} \quad (23)$$

This equation is of the same form as Gauss's law for gravity, so everything discussed previously for gravity also applies here. Although this equation is true in general, it has a good practical use for easily calculating the electric field E due to a point, sphere, line, cylinder, or plane of electric charge. To do this, you do just as we did with the gravity examples: draw an imaginary Gaussian surface around the charge q , write down Gauss's Law, evaluate the integral, and solve for the electric field E . Here q is the total electric charge enclosed by S . The electric field \mathbf{E} points away from positive electric charge, and toward negative charge. The constant ϵ_0 is called the *permittivity of free space*, and has a value of $8.854187817 \times 10^{-12}$ F/m.

You'll find more details about Maxwell's equations in General Physics II.